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Robert M. Freund

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Robert M. Freund[2]
M.I.T.

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Abstract

For a conic optimization problem

$$\begin{array}{lll} P: & \text{minimize}_x & c^T x \\ & \text{s.t.} & Ax = b \\ & & x \in C \end{array}$$

and its dual:

$$\begin{array}{lll} D: & \text{supremum}_{y,s} & b^T y \\ & \text{s.t.} & A^T y + s = c \\ & & s \in C^*, \end{array}$$

we present a geometric relationship between the maximum norms of the level sets of the primal and the inscribed sizes of the level sets of the dual (or the other way around).

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[2]MIT Sloan School of Management, 50 Memorial Drive, Cambridge, MA 02142-1347, USA. email: rfreund@mit.edu

1 Primal-Dual Geometry of Level Sets for Linear Optimization

Consider the following dual pair of linear optimization problems:

$$\begin{array}{lll} LP: & \text{minimize} & c^T x \\ & \text{s.t.} & Ax = b \\ & & x \geq 0, \end{array}$$

and:

$$\begin{array}{lll} LD: & \text{maximize} & b^T y \\ & \text{s.t.} & A^T y + s = c \\ & & s \geq 0\ , \end{array}$$

whose common optimal value is z^*. For $\epsilon > 0$ and $\delta > 0$, define the ϵ- and δ-level sets for the primal and dual problems as follows::

$$P_\epsilon := \left\{ x \mid Ax = b, x \geq 0, c^T x \leq z^* + \epsilon \right\}$$

and

$$D_\delta := \left\{ s \mid A^T y + s = c, s \geq 0 \text{ for some } y \text{ satisfying } b^T y \geq z^* - \delta \right\}\ .$$

Define:

$$\begin{array}{rll} R_\epsilon := & \max & \|x\|_1 \\ & \text{s.t.} & Ax = b \\ & & c^T x \leq z^* + \epsilon \\ & & x \geq 0 \end{array} \tag{1}$$

and

$$\begin{array}{rll} r_\delta := & \max & \min_j\{s_j\} \\ & \text{s.t.} & A^T y + s = c \\ & & b^T y \geq z^* - \delta \\ & & s \geq 0 \end{array} \tag{2}$$

The quantity R_ϵ is simply the size of the largest vector x in the primal level set P_ϵ, measured in the L_1-norm. The quantity r_δ can be interpreted as the positivity of the most positive vector s in the dual level set D_δ, or equivalently as the maximum distance to the boundary of the nonnegative

orthant over all points s in D_δ. The following theorem presents a reciprocal relationship between R_ϵ and r_δ:

Theorem 1.1 *Suppose that z^* is finite. If R_ϵ is positive and finite, then*

$$\min\{\epsilon, \delta\} \;\le\; R_\epsilon \cdot r_\delta \;\le\; \epsilon + \delta \; .$$

Otherwise, $R_\epsilon = 0$ if and only if $r_\delta = +\infty$, and $R_\epsilon = +\infty$ if and only if $r_\delta = 0$.

The proof of this theorem follows as a special case of a more general result for convex conic optimization, namely Lemma 4.1 in Section 4.

Theorem 1.1 bounds the size of the largest vector in P_ϵ and the positivity of the most positive vector in D_δ from above and below, and shows that these quantities are almost exactly inversely proportional. In fact, taking $\delta = \epsilon$, the result states that $R_\epsilon \cdot r_\epsilon$ lies in the interval $[\epsilon, 2\epsilon]$.

Corollary 1.1 *If $R_\epsilon < \infty$, then*

$$R_{\epsilon'} \le \left(\frac{\epsilon + \epsilon'}{\epsilon}\right) R_\epsilon \tag{3}$$

for all $\epsilon' \ge \epsilon$. ∎

Proof: If $R_\epsilon = 0$ the result follows trivially, since then $R_{\epsilon'} = 0$ for all $\epsilon' > 0$. So suppose that $0 < R_\epsilon < +\infty$. Let $\delta := \epsilon$. Then from Theorem 1.1 we have $\epsilon \le R_\epsilon \cdot r_\epsilon \le 2\epsilon$ and $\epsilon \le R_{\epsilon'} \cdot r_\epsilon \le \epsilon' + \epsilon$, and so $R_{\epsilon'} \le \left(\frac{\epsilon+\epsilon'}{\epsilon}\right) R_\epsilon$. ∎

Corollary 1.1 bounds the rate of growth of $R_{\epsilon'}$ as ϵ' increases, and shows that $R_{\epsilon'}$ grows at most linearly in ϵ' and at a rate no greater than $\frac{R_\epsilon}{\epsilon}$.

Of course, there is a version of (3) for r_δ and $r_{\delta'}$, namely:

$$r_{\delta'} \ge \left(\frac{\delta'}{\delta}\right) r_\delta \tag{4}$$

for all $0 \le \delta' \le \delta$, which is true as an elementary consequence of the convexity of the feasible region of LD (independent of the results contained herein). However, (3) does not seem to lend itself to an independent elementary proof.

By exchanging the roles of the primal and dual problems, we obviously can construct analogous results for the most positive vector x in P_ϵ as well as for the size of the largest vector s in D_δ.

2 Conic Optimization with a Norm on X

We now consider the generalization of linear optimization to convex optimization in conic linear form:

$$\begin{array}{lll} P: \; z^* := \text{minimum}_x & c^T x \\ \qquad\qquad \text{s.t.} & Ax = b \\ & x \in C, \end{array}$$

and its dual:

$$\begin{array}{lll} D: \; v^* := \text{supremum}_{y,s} & b^T y \\ \qquad\qquad \text{s.t.} & A^T y + s = c \\ & s \in C^*, \end{array}$$

where $C \subset X$ is a closed convex cone in the (finite) n-dimensional linear vector space X, and b lies in the (finite) m-dimensional vector space Y. This format for convex optimization dates back at least to Duffin [2]. Strong duality results can be found in [2] as well as in Ben-Israel et al. [1].

For $\epsilon > 0$ and $\delta > 0$, we define the ϵ- and δ-level sets for the primal and dual problems as follows:

$$P_\epsilon := \left\{ x \mid Ax = b, x \in C, c^T x \leq z^* + \epsilon \right\}$$

and

$$D_\delta := \left\{ s \mid A^T y + s = c, s \in C^* \text{ for some } y \in Y^* \text{ satisfying } b^T y \geq v^* - \delta \right\} .$$

We make the following assumption:

Assumption A: z^* and v^* are both finite. The cone C contains no line (and so C^* has an interior).

Suppose that X is endowed with a norm $\|\cdot\|$ and so X^* is endowed with the dual norm $\|\cdot\|_*$. Let $B(v, r)$ and $B^*(w, r)$ denote the balls of radius r centered at $v \in X$ and $w \in X^*$, respectively, defined for the appropriate norms.

We denote the maximum norm of P_ϵ by R_ϵ, defined as:

$$\begin{array}{lll} R_\epsilon := & \max_x & \|x\| \\ & \text{s.t.} & x \in P_\epsilon \, . \end{array} \tag{5}$$

We denote by r_δ the inscribed size of D_δ, defined as:

$$
\begin{array}{rll}
r_\delta := & \max_{s,r} & r \\
& \text{s.t.} & s \in D_\delta \\
& & B^*(s,r) \subset C^* \ .
\end{array} \tag{6}
$$

As in the case of linear optimization, r_δ measures the distance of the most interior point of the dual level set D_δ to the boundary of the cone C^*.

Before presenting the version of Theorem 1.1 for convex conic optimization, we first review the concept of the min-width of a regular cone, see [3]. A cone K is *regular* if K is a closed convex cone, has a nonempty interior, and is pointed (i.e., contains no line). We use the following definition of the min-width of K:

Definition 2.1 *If K is a regular cone in the normed linear vector space X, the min-width of K is defined as:*

$$\tau_K := \max_{x \in \text{int}K} \left\{ \frac{\text{dist}\,(x, \partial K)}{\|x\|} \right\} = \max_{x \in \text{int}K} \left\{ \frac{r}{\|x\|} \mid B(x,r) \subset K \right\} .$$

We remark that τ_K measures the maximum ratio of the radius to the norm of the center of an inscribed ball in K, and so larger values of τ_K correspond to an intuitive notion of greater minimum width of K. Note that $\tau_K \in (0,1]$ if K is a regular cone, since K has a nonempty interior and K is pointed, and τ_K is attained for some $x^0 \in \text{int}K$ satisfying $\|x^0\| = 1$, as well as along the ray αx^0 for all $\alpha > 0$. Let τ_{K^*} be defined similarly for the dual cone K^*.

The following is analogous to Theorem 1.1 for conic problems:

Theorem 2.1 *Suppose that Assumption A holds. If R_ϵ is positive and finite, then $z^* = v^*$ and*

$$\tau_{C^*} \cdot \min\{\epsilon, \delta\} \leq R_\epsilon \cdot r_\delta \leq \epsilon + \delta \ .$$

Otherwise, $R_\epsilon = 0$ if and only if $r_\delta = +\infty$, and $R_\epsilon = +\infty$ if and only if $r_\delta = 0$.

Here we have had to introduce the min-width τ_{C^*} into the left inequality, somewhat weakening the result. In the next section we show how to define a family of cone-based norms for which $\tau_{C^*} = 1$; we also show that for norms induced by a ϑ-normal barrier function on C that $\tau_{C^*} \geq 1/\sqrt{\vartheta}$. Theorem 2.1 is proved in Section 4.

Corollary 2.1 *If $R_\epsilon < \infty$, then*

$$R_{\epsilon'} \leq \left(\frac{\epsilon + \epsilon'}{\epsilon}\right)\left(\frac{1}{\tau_{C^*}}\right) R_\epsilon$$

for all $\epsilon' \geq \epsilon$.

Proof: If $R_\epsilon = 0$ the result follows trivially, since then $R_{\epsilon'} = 0$ for all $\epsilon' > 0$. So suppose that $0 < R_\epsilon < +\infty$. Let $\delta := \epsilon$. Then from Theorem 2.1 we have $\tau_{C^*} \cdot \epsilon \leq R_\epsilon \cdot r_\epsilon \leq 2\epsilon$ and $\tau_{C^*} \cdot \epsilon \leq R_{\epsilon'} \cdot r_\epsilon \leq \epsilon' + \epsilon$, and so $R_{\epsilon'} \leq \left(\frac{\epsilon+\epsilon'}{\epsilon}\right)\left(\frac{1}{\tau_{C^*}}\right) R_\epsilon$.
∎

3 Min-widths of Two Natural Families of Norms

In this section we assume throughout that C is a regular cone.

3.1 Min-widths for the Family of Norms induced by a ϑ-normal Barrier

Suppose that $F(\cdot) : \mathrm{int}C \to \Re$ is a ϑ-normal barrier for C, see [5]. Then $F^*(\cdot) : \mathrm{int}C^* \to \Re$, the conjugate function of $F(\cdot)$, is also a ϑ-normal barrier for C^*, see [5] as well.

Let $s^0 \in \mathrm{int}C^*$ be given. The norm induced by the ϑ-normal barrier $F(\cdot)$ at s^0 is defined as follows:

$$\|v\|_{*,s^0} := \sqrt{v^T H^*(s^0) v}\ ,$$

where $H^*(s^0)$ is the Hessian of $F^*(\cdot)$ evaluated at $s = s^0$. It then follows from Theorem 2.1.1 of [5] that $B^*(s^0, 1) \subset C^*$ and from Proposition 2.3.4 of [5] that $\|s^0\|_{*,s^0} = \sqrt{\vartheta}$. Therefore under the dual norm $\|v\|_* := \|v\|_{*,s^0}$ we have $\tau_{C^*} \geq 1/\sqrt{\vartheta}$.

3.2 A Family of Norms on X for which $\tau_{C^*} = 1$

For every $s^0 \in \mathrm{int}C^*$, there is a norm analogous to the L_∞-norm for the nonnegative orthant, for which the associated min-width is $\tau_{C^*} = 1$. To see this, consider a given interior point $s^0 \in \mathrm{int}C^*$, and define the following norm:

$$\begin{array}{rll} \|s\|_* := & \min_\alpha & \alpha \\ & \text{s.t.} & s + \alpha s^0 \in C^* \\ & & -s + \alpha s^0 \in C^* \end{array}$$

whose dual norm is easily seen to be:

$$\begin{array}{rll} \|x\| := & \min_{x^1, x^2} & (s^0)^T(x^1 + x^2) \\ & \text{s.t.} & x^1 - x^2 = x \\ & & x^1 \in C \\ & & x^2 \in C \ . \end{array}$$

Under $\|\cdot\|_*$, it is easily shown that $\|s^0\|_* = 1$ and $\tau_{C^*} = 1$.

In the case when $X = \Re^n$, $C = C^* = \Re^n_+$, and $s^0 = e$, we recover the L_∞-norm as $\|s\|_*$ for $s \in X = \Re^n$ and the L_1-norm as $\|x\|$ for $x \in X = \Re^n$.

4 A Technical Lemma, and Proofs of Main Results

The technical result that is the basis for the other results in this paper is as follows. Let $s^0 \in \text{int}C^*$ be given. Let

$$\begin{array}{rll} \bar{R}_\epsilon := & \max_x & (s^0)^T x \\ & \text{s.t.} & x \in P_\epsilon \end{array} \qquad (7)$$

and

$$\begin{array}{rll} \bar{r}_\delta := & \max_{s,r} & r \\ & \text{s.t.} & s \in D_\delta \\ & & s - r s^0 \in C^* \ . \end{array} \qquad (8)$$

Lemma 4.1 *Suppose that Asssumption A holds. If $\bar{R}_\epsilon$ is positive and finite, then $z^* = v^*$ and*

$$\min\{\epsilon, \delta\} \ \le\ \bar{R}_\epsilon \cdot \bar{r}_\delta \ \le\ \epsilon + \delta \ . \qquad (9)$$

Otherwise, $\bar{R}_\epsilon = 0$ if and only if $\bar{r}_\delta = +\infty$, and $\bar{R}_\epsilon = +\infty$ if and only if $\bar{r}_\delta = 0$.

Before proving Lemma 4.1, we first prove strong duality for P and D when $\bar{R}_\epsilon < +\infty$.

Proposition 4.1 *If $\bar{R}_\epsilon < +\infty$, then $z^* = v^*$.*

Proof: Note that if $\bar{R}_\epsilon < \infty$ for some $\epsilon > 0$, then all of the level sets P_ϵ of P are bounded, and so P attains its minimum. The boundedness of P_ϵ also implies that

$$\{0\} = \{v \in X \mid Av = 0, v \in C, c^T v \leq 0\} \ . \tag{10}$$

It is elementary to show that $z^* \geq v^*$. Suppose that $z^* > v^*$, let $\bar{\epsilon} = \frac{z^* - v^*}{2}$, and let

$$S = \{(w, \alpha) | \exists y \in Y^*, s \in C^* \text{ satisfying } w = A^T y + s - c, b^T y \geq v^* + \bar{\epsilon} - \alpha\}.$$

Then S is a nonempty convex set in $X^* \times \Re$, and $(0,0) \notin S$, whereby there exists $(x, \theta) \neq 0$ satisfying $x^T w + \theta\alpha \geq 0$ for all $(w, \alpha) \in S$. Therefore:

$$x^T \left(A^T y + s - c\right) + \theta\left(-b^T y + v^* + \bar{\epsilon} + \eta\right) \geq 0 \quad \forall y \in Y^*, \forall s \in C^*, \forall \eta \geq 0 \ . \tag{11}$$

This implies that $Ax = b\theta$, $\theta \geq 0$, and $x \in C$. We now have two cases:

Case 1: $\theta > 0$. Without loss of generality we can assume that $\theta = 1$. Therefore x is feasible for P, and (11) also implies that $z^* \leq c^T x \leq v^* + \bar{\epsilon} < z^*$, which is a contradiction.

Case 2: $\theta = 0$. In this case $x \neq 0$, $x \in C$, $Ax = 0$, and (11) implies that $c^T x \leq 0$, contradicting (10).

In both cases we have a contradiction, and so $z^* = v^*$. ▮

Proof of Lemma 4.1: Let us first examine the extreme cases $\bar{r}_\delta = +\infty$ and $\bar{r}_\delta = 0$. If $\bar{r}_\delta = +\infty$, then it is a straightforward exercise in convex analysis to show under Assumption A that $z^* = v^* = 0, b = 0$, the feasible region of P is the singleton $\{0\}$, and there exists $(\hat{y}, \hat{s})$ satisfying $A^T\hat{y} + \hat{s} = 0, \hat{s} \in \text{int}C^*$. This then implies that $\bar{R}_\epsilon = 0$. If $\bar{r}_\delta = 0$, then it is also a straightforward exercise in convex analysis to show under Assumption A that there exists $\hat{x} \neq 0$ satisfying $A\hat{x} = 0, \hat{x} \in C, c^T\hat{x} \leq 0$. This then implies that $\bar{R}_\epsilon = +\infty$.

We now consider the case when $0 < \bar{r}_\delta < +\infty$. For any $\alpha \in (0, \bar{r}_\delta)$, there exists $\hat{s} \in D_\delta$ satisfying $v := \hat{s} - \alpha s^0 \in C^*$. For every $x \in P_\epsilon$ we have

$$\epsilon + \delta \geq \hat{s}^T x = (v + \alpha s^0)^T x \geq \alpha (s^0)^T x,$$

whereby $\bar{R}_\epsilon \leq \frac{\epsilon+\delta}{\alpha}$, and so $\bar{R}_\epsilon \bar{r}_\delta \leq \epsilon + \delta$. This proves the second inequality of (9).

To prove the first inequality of (9), we proceed as follows. We write (7) as the following program together with its dual:

$$\begin{array}{llll} \bar{P}: & \bar{R}_\epsilon := \max_x & (s^0)^T x \\ & \text{s.t.} & Ax = b \\ & & c^T x \leq z^* + \epsilon \\ & & x \in C \end{array} \qquad \begin{array}{llll} \bar{D}: & \bar{v} := \inf_{y,s,\theta} & -b^T y + (z^* + \epsilon)\theta \\ & \text{s.t.} & A^T y + s = \theta c \\ & & s - s^0 \in C^* \\ & & \theta \geq 0. \end{array}$$

In Lemma 4.2 below, we show strong duality for $\bar{P}$ and $\bar{D}$, namely $\bar{R}_\epsilon = \bar{v}$, under Assumption A and in the case when $\bar{R}_\epsilon < +\infty$.

Because the feasible region of $\bar{P}$ is bounded, $\bar{P}$ will attain its optimum. For $\alpha \in (0, \min\{\epsilon, \delta\})$ and sufficiently small, let (y, s, θ) be a feasible solution of $\bar{D}$ satisfying

$$-b^T y + (z^* + \epsilon)\theta \leq \bar{R}_\epsilon + \alpha \tag{12}$$

and define $v := s - s^0 \in C^*$. We will show below that

$$\bar{R}_\epsilon \bar{r}_\delta \geq \frac{\bar{R}_\epsilon}{\bar{R}_\epsilon + \alpha} \left(\min\{\epsilon, \delta - \alpha\}\right) \tag{13}$$

and letting $\alpha \to 0$ establishes the first inequality of (9). We prove (13) by considering three cases.

Case 1: $\theta = 0$. In this case $A^T y + s = 0$ and $-b^T y \leq \bar{R}_\epsilon + \alpha$. Let $(\bar{y}, \bar{s})$ be any feasible solution of D satisfying $b^T \bar{y} \geq z^* - \alpha$, and define

$$(\hat{y}, \hat{s}) := (\bar{y}, \bar{s}) + \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha}(y, s).$$

Then $(\hat{y}, \hat{s})$ is feasible for D, and

$$b^T \hat{y} = b^t \bar{y} + \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha} b^T y \geq z^* - \alpha - \delta + \alpha = z^* - \delta.$$

Also, $\hat{s} - \frac{\delta-\alpha}{\bar{R}_\epsilon+\alpha} s^0 = \frac{\delta-\alpha}{\bar{R}_\epsilon+\alpha} v + \bar{s} \in C^*$, whereby $\hat{s} \in D_\delta$ and $\bar{r}_\delta \geq \frac{\delta-\alpha}{\bar{R}_\epsilon+\alpha}$. This then implies (13).

Case 2: $\theta > 0$ and $\frac{\bar{R}_\epsilon+\alpha}{\theta} - \epsilon \leq \delta$. Define

$$(\hat{y}, \hat{s}) = \frac{1}{\theta}(y, s),$$

whereby $(\hat{y}, \hat{s})$ satisfies $\hat{s} \in C^*$, $A^T\hat{y} + \hat{s} = c$, and

$$b^T\hat{y} = \frac{1}{\theta}b^Ty \geq -\frac{\bar{R}_\epsilon + \alpha}{\theta} + z^* + \epsilon \geq z^* - \delta,$$

which shows that $\hat{s} \in D_\delta$. Furthermore, $\hat{s} = \frac{s^0}{\theta} + \frac{v}{\theta}$, $v \in C^*$, and so $\hat{s} - \frac{1}{\theta}s^0 \in C^*$, whence $\bar{r}_\delta \geq \frac{1}{\theta}$. However,

$$z^* \geq b^T\hat{y} \geq -\frac{\bar{R}_\epsilon + \alpha}{\theta} + z^* + \epsilon,$$

and so $\frac{1}{\theta} \geq \frac{\epsilon}{\bar{R}_\epsilon + \alpha}$, whereby $\bar{r}_\delta \geq \frac{1}{\theta} \geq \frac{\epsilon}{\bar{R}_\epsilon + \alpha}$, which then implies (13).

Case 3: $\theta > 0$ and $\frac{\bar{R}_\epsilon + \alpha}{\theta} - \epsilon \geq \delta$. Let $(\bar{y}, \bar{s})$ be any feasible solution of D satisfying

$$b^T\bar{y} \geq z^* - \alpha \tag{14}$$

and define

$$(\hat{y}, \hat{s}) = \lambda\left(\frac{(y, s)}{\theta}\right) + (1 - \lambda)(\bar{y}, \bar{s})$$

where

$$\lambda = \frac{\delta - \alpha}{\frac{\bar{R}_\epsilon + \alpha}{\theta} - \epsilon - \alpha}.$$

Then $\lambda \in [0, 1]$ for $\alpha \in (0, \delta)$, and so $(\hat{y}, \hat{s})$ is a convex combination of $\frac{(y,s)}{\theta}$ and $(\bar{y}, \bar{s})$ and so satisfies $A^T\hat{y} + \hat{s} = c$, $\hat{s} \in C^*$. It also follows from (12) and (14) that $b^T\hat{y} \geq z^* - \delta$, whereby $\hat{s} \in D_\delta$. Finally, $\hat{s} - \frac{\lambda}{\theta}s^0 \in C^*$, and so

$$\bar{r}_\delta \geq \frac{\lambda}{\theta} = \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha - \alpha\theta - \epsilon\theta} \geq \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha},$$

from which (13) follows.

Therefore (13) is true, and letting $\alpha \to 0$ proves the first inequality of (9). ∎

Lemma 4.2 *Suppose that Assumption A holds. If $\bar{R}_\epsilon$ is positive and finite, then $\bar{R}_\epsilon = \bar{v}$.*

Proof: Note that for x and (y, s, θ) feasible for $\bar{P}$ and $\bar{D}$, respectively,

$$0 \leq x^T(s - s^0) = \theta c^Tx - y^TAx - (s^0)^Tx \leq \theta(z^* + \epsilon) - y^Tb - (s^0)^Tx,$$

and so $(s^0)^T x \le -y^T b + (z^* + \epsilon)\theta$. Therefore $\bar{R}_\epsilon \le \bar{v}$. Suppose that $\bar{R}_\epsilon < \bar{v}$, let $\bar{\epsilon} = \frac{\bar{v} - \bar{R}_\epsilon}{2}$ and let

$$S = \{ \quad (w, \alpha) \mid \exists y \in Y^*, s \in X^*, \theta \ge 0 \text{ satisfying} \\ w = -\theta c + A^T y + s, -b^T y + (z^* + \epsilon)\theta \le \bar{v} - \bar{\epsilon} + \alpha, s - s^0 \in C^* \}.$$

Then S is a nonempty convex set in $X^* \times \Re$, and $(0,0) \notin S$, whereby there exists $(x, \beta) \ne 0$ satisfying $x^T w + \beta\alpha \ge 0$ for all $(w, \alpha) \in S$. Therefore

$$x^T \left(-\theta c + A^T y + s^0 + v\right) + \beta\left(-b^T y + (z^* + \epsilon)\theta - \bar{v} + \bar{\epsilon} + \eta\right) \ge 0$$
$$\forall y \in Y^*, \quad \forall v \in C^*, \quad \forall \theta \ge 0, \quad \forall \eta \ge 0 \,. \tag{15}$$

This implies that $x \in C$, $Ax - b\beta = 0$, and $\beta \ge 0$. We now have two cases:

Case 1: $\beta > 0$. Without loss of generality we can assume that $\beta = 1$. Therefore x is feasible for P and (15) also implies that $(s^0)^T x \ge \bar{v} - \bar{\epsilon} > \bar{R}_\epsilon$, which is a contradiction.

Case 2: $\beta = 0$. In this case (15) implies that $Ax = 0$, $x \in C$, $c^T x \le 0$, and also $x \ne 0$, which contradicts (10).

In both cases we obtain a contradiction, and so $\bar{R}_\epsilon = \bar{v}$. ▮

Proof of Theorem 1.1: Follows from Lemma 4.1 by setting $C = \Re^n_+$ and $s^0 = e := (1, \ldots, 1)^T$. ▮

Before proving Theorem 2.1, we state the following two propositions. The proof of the Proposition 4.2 follows immediately from Proposition 3 of [4] and Proposition 2.1 of [3]. Proposition 4.3 is a special case of the Hahn-Banach Theorem; for a short proof of this proposition based on the subdifferential operator, see Proposition 2 of [4].

Proposition 4.2 *Suppose K^* is a regular convex cone whose min-width τ_{K^*} is attained at some point $s^0 \in intK^*$ satisfying $\|s^0\|_* = 1$. Then*

$$\tau_{K^*}\|x\| \;\le (s^0)^T x \;\le \|x\|$$

for all $x \in K$. ▮

Proposition 4.3 *For every $x \in X$, there exists $\bar{x} \in X^*$ with the property that $\|\bar{x}\|_* = 1$ and $\|x\| = \bar{x}^T x$.* ∎

Proof of Theorem 2.1: Let $s^0 \in C^*$ be a point in C^* at which the min-width of C^* is attained, normalized so that $\|s^0\|_* = 1$. Then $B^*(s^0, \tau_{C^*}) \subset C^*$ and also $\tau_{C^*}\|x\| \le (s^0)^T x \le \|x\|$ for all $x \in C$ (from Proposition 4.2). From (7) and (5) it follows that $\tau_{C^*} R_\epsilon \le \bar{R}_\epsilon \le R_\epsilon$. Also, since the constraint $s - rs^0 \in C^*$ implies that $B^*(s, r\tau_{C^*}) \subset C^*$ and the constraint $B^*(s,r) \subset C^*$ implies that $s - rs^0 \in C^*$, it follows from (8) and (6) that $\tau_{C^*}\bar{r}_\delta \le r_\delta \le \bar{r}_\delta$. Therefore $R_\epsilon = 0$ if and only if $\bar{R}_\epsilon = 0$ if and only if $\bar{r}_\delta = +\infty$ if and only if $r_\delta = +\infty$, and $R_\epsilon = +\infty$ if and only if $\bar{R}_\epsilon = +\infty$ if and only if $\bar{r}_\delta = 0$ if and only if $r_\delta = 0$, from Lemma 4.1.

If R_ϵ is positive and finite, it follows then that $R_\epsilon \cdot r_\delta \ge \tau_{C^*}\bar{R}_\epsilon \cdot \bar{r}_\delta \ge \tau_{C^*} \min\{\epsilon, \delta\}$, from Lemma 4.1, which proves one of the inequalities of the theorem. Also, this shows that $r_\delta > 0$. For any $x \in P_\epsilon$, there exists $\bar{x} \in X^*$ for which $\|\bar{x}\|_* = 1$ and $\bar{x}^T x = \|x\|$, see Proposition 4.3. And for any $\alpha \in (0, r_\delta)$, there exists $(\hat{y}, \hat{s})$ feasible for D such that $\hat{s} \in D_\delta$ and $B^*(\hat{s}, \alpha) \subset C^*$. Then $\epsilon + \delta \ge c^T x - b^T \hat{y} = \hat{s}^T x = (\hat{s} - \alpha\bar{x} + \alpha\bar{x})^T x \ge \alpha\bar{x}^T x = \alpha\|x\|$, where the second inequality follows from the fact that $(\hat{s} - \alpha\bar{x}) \in C^*$ since $B^*(\hat{s}, \alpha) \subset C^*$. Therefore $\|x\| \le \frac{\epsilon+\delta}{\alpha}$ for any $\alpha \in (0, r_\delta)$ and $x \in P_\epsilon$, whereby $R_\delta \cdot r_\delta \le \epsilon + \delta$, completing the proof. ∎

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